1. Assignment 1

1.1. Problem 1. In the context of a poset, use upper sets to prove that, when it exists, the join $\vee$ satisfies the following three properties.

(a) $(x \vee y) \vee z = x \vee (y \vee z)$ *(associativity)*
(b) $x \vee y = y \vee x$ *(commutativity)*
(c) $x \vee x = x$ *(idempotency)*
(d) Use duality to prove that the above also hold for $\wedge$.

These three properties respectively imply that bracketing, order, and multiple incidence don't affect joins and meets. Hence, for a lattice $(P, \preceq)$ and a *finite* subset $S \subseteq P$, we can construct suprema and infima in terms of joins and meets:

$$\sup S = \bigvee_{s \in S} s \qquad \inf S = \bigwedge_{s \in S} s.$$

(e) Show that, in an infinite lattice, $\sup S$ and $\inf S$ need not exist for infinite S.

Suppose there exist elements $\top$ and $\bot$ that satisfy the following.

$$x \preceq \top \text{ for all } x \in P$$
$$\bot \preceq x \text{ for all } x \in P$$

We call $\top$ the *top* element and $\bot$ the *bottom* element.

(f) Prove that an equivalent definition for $\top$ and $\bot$ is that they satisfy *unitality*:

$$x \vee \bot = x \text{ for all } x \in P$$
$$x \wedge \top = x \text{ for all } x \in P$$

(g) Use this to show that $\top, \bot$ can also be defined as suprema and infima:

$$\top = \inf \varnothing \qquad \bot = \sup \varnothing$$

(h) Prove that, when P is a finite lattice, $\top$ and $\bot$ can be computed as follows.

$$\top = \sup P \qquad \bot = \inf P$$

(i) Show that, in an infinite lattice, $\top$ and $\bot$ need not exist.

1.2. Problem 2. Let $\mathbb{P}_n = p_0 \cdot p_1 \cdots p_{n-1}$, be the product of the first n primes. Construct an explicit isomorphism

$$(\mathcal{P}\mathbf{n}, \subseteq) \xrightarrow{\cong} (\langle \mathbb{P}_n \rangle, |),$$

i.e. give rules for monotone maps f, g of type

$$f : \mathcal{P}\mathbf{n} \to \langle \mathbb{P}_n \rangle$$
$$g : \langle \mathbb{P}_n \rangle \to \mathcal{P}\mathbf{n}$$

and demonstrate that $g \circ f = \mathbb{1}_{\mathcal{P}\mathbf{n}}$ and $f \circ g = \mathbb{1}_{\langle \mathbb{P}_n \rangle}$.

1.3. Problem 3.

(a) Prove that

$$\mathrm{cof} : \langle n \rangle^{\mathrm{op}} \to \langle n \rangle :: a \mapsto \tfrac{n}{a}$$

forms a Galois connection with its opposite.

(b) Express the De Morgan laws for this Galois connection.
(c) Select n in the above De Morgan laws so as to deduce the famous identity

$$a \cdot b = \gcd(a, b) \cdot \mathrm{lcm}(a, b).$$

1.4. **Problem 4.** Let $f : X \to Y$ be a map and $f^* : \mathcal{P}Y \to \mathcal{P}X$ be its preimage. Define the *direct image* as:

$$f_* : \mathcal{P}X \to \mathcal{P}Y :: S \mapsto \{f(s) \mid s \in S\}$$

(a) Show that f_* is monotone.
(b) Show that (f_*, f^*) is a Galois connection.
(c) Write the two de Morgan laws implied by the above Galois connection.
(d) Provide a map $f : X \to Y$ for which there exist $A, B \subseteq X$ such that

$$f_*(A \cap B) \neq f_*(A) \cap f_*(B).$$

(e) Prove directly that, in contrast

$$f^*(A \cup B) = f^*(A) \cup f^*(B).$$

Lament this asymmetry. Now define the *indirect image* as:

$$f_! : \mathcal{P}X \to \mathcal{P}Y :: S \mapsto \{y \in Y \mid f^*(y) \subseteq S\}.$$

(f) Show that $f_!$ is monotone.
(g) Show that $(f^*, f_!)$ is a Galois connection.
(h) Write the corresponding de Morgan laws. What do you notice?
(i) Provide a map $f : X \to Y$ for which there exist $A, B \subseteq X$ such that

$$f_!(A \cup B) \neq f_!(A) \cup f_!(B).$$

(j) Show that the direct image f_* is equal to the map below

$$f_\exists : \mathcal{P}X \to \mathcal{P}Y :: S \mapsto \{y \in Y \mid \exists x \in f^*(y), x \in S\}.$$

(k) Show that the indirect image $f_!$ is equal to the map below

$$f_\forall : \mathcal{P}X \to \mathcal{P}Y :: S \mapsto \{y \in Y \mid \forall x \in f^*(y), x \in S\}.$$

(l) Discuss the higher symmetry that resolved what seemed to be an asymmetry.