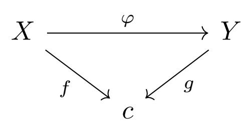
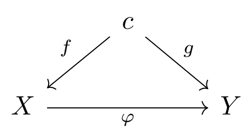
2. Assignment 2

2.1. **Problem 1.** For any category $\mathcal{C}$ and object $c \in |\mathcal{C}|$, prove that there is a category, which we denote by $\mathcal{C}/c$ and call the *slice category over* c, whose objects are arrows $f : X \to c$ with codomain c and morphisms $[f : X \to c] \to [g : Y \to c]$ are arrows $\varphi : X \to Y$ for which the following triangle commutes.



Analogously, show that there is a category, which we denote by $c/\mathcal{C}$ and call the *slice category under* c, whose objects are arrows $f : c \to X$ with domain c and morphisms $[f : c \to X] \to [g : c \to Y]$ are arrows $\varphi : X \to Y$ for which the following triangle commutes.



Prove that $c/\mathcal{C}$ is the same as $[\mathcal{C}^{\text{op}}/c]^{\text{op}}$.

2.2. **Problem 2.** An arrow $f : X \to Y$ is said to be a *monomorphism* if for all pairs $g, g' : W \to X$, the following condition holds:

$$fg = fg' \Rightarrow h = h'.$$

Dually, $f : X \to Y$ is said to be an *epimorphism* if for all $h, h' : Y \to Z$:

$$hf = h'f \Rightarrow h = h'.$$

Prove that $f : X \to Y$ in **Set** is monomorphism if and only if f is injective, and an epimorphism if and only if f is surjective.

2.3. **Problem 3.** Show that, given a category $\mathcal{C}$, there is a functor

$$\mathcal{C}(-,-) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathbf{Set}$$

given by the rules

- $(X, Y) \mapsto \mathcal{C}(X, Y)$ on an object (X, Y)
- $\mathcal{C}(f, g) : \mathcal{C}(X, Y) \to \mathcal{C}(W, Z) :: \varphi \mapsto f /\!\!/ \varphi /\!\!/ g$ on an arrow $(W \xrightarrow{f} X, Y \xrightarrow{g} Z)$.

2.4. **Problem 4.** Consider the category $\mathbb{2} = 0 \to 1$. Let $F_0, F_1 : \mathcal{C} \to \mathcal{D}$ be functors and define the inclusion functors $\mathcal{C}_i : \mathcal{C} \to \mathcal{C} \times \mathbb{2} :: X \mapsto (X, i)$ for $i = 1, 2$. Let $\mathcal{T} : \mathcal{C} \times \mathbb{2} \to \mathcal{D}$ be a functor such that $\mathcal{T} \circ \mathcal{C}_i = F_i$. Prove that $\mathcal{T}$ gives a natural transformation $F_0 \to F_1$. In turn, given a natural transformation $T : F_0 \to F_1$, prove that it can be recast as a functor $\mathcal{C} \times \mathbb{2} \to \mathcal{D}$.

2.5. **Problem 5.**

(a) Let $(P, \preceq), (Q, \sqsubseteq)$ be preorders. Prove that $\mathbf{Pre}(P, Q)$ is a preorder.
(b) If $(P, \preceq), (Q, \sqsubseteq)$ are posets, prove that $\mathbf{Pos}(P, Q)$ is a poset.
(c) If $(P, \preceq), (Q, \sqsubseteq)$ are total orders, prove that $\mathbf{Tot}(P, Q)$ is *not* a total order.
(d) Show that any set map $X \to P$ is automatically a monotonic map $\mathbf{D}X \to P$.

(e) Let $\mathbb{B}$ have the posetal structure given by the relation $\bot \preceq \top$. Given two maps $\mathbf{P}, \mathbf{Q} : \mathbf{D}X \to \mathbb{B}$, interpreted as predicates (i.e. conditions), describe the logical interpretation of the existence of a natural transformation $\mathbf{P} \to \mathbf{Q}$.
(f) Describe the join, meet, top, and bottom in the poset $[X; \mathbb{B}] := \mathbf{Pos}(\mathbf{D}X, \mathbb{B})$.
(g) Prove the monotonicity of the map $\iota : \mathbb{B} \to [X; \mathbb{B}] :: * \mapsto \lambda x.*$.
(h) Prove the monotonicity of the map $\exists : [X; \mathbb{B}] \to \mathbb{B} :: \mathbf{P} \mapsto \begin{cases} \top & \exists x, \mathbf{P}(x) = \top \\ \bot & \text{otherwise} \end{cases}$
(i) Prove that $(\exists, \iota)$ is a Galois connection.
(j) This guarantees that $\exists(\mathbf{P} \vee \mathbf{Q}) = \exists\mathbf{P} \vee \exists\mathbf{Q}$. Explain why $\exists(\mathbf{P} \wedge \mathbf{Q}) \neq \exists\mathbf{P} \wedge \exists\mathbf{Q}$.
(k) Prove the monotonicity of the map $\forall : [X; \mathbb{B}] \to \mathbb{B} :: \mathbf{P} \mapsto \begin{cases} \top & \forall x, \mathbf{P}(x) = \top \\ \bot & \text{otherwise} \end{cases}$
(l) Prove that $(\iota, \forall)$ is a Galois connection.
(m) This guarantees that $\forall(\mathbf{P} \wedge \mathbf{Q}) = \forall\mathbf{P} \wedge \forall\mathbf{Q}$. Explain why $\forall(\mathbf{P} \vee \mathbf{Q}) \neq \forall\mathbf{P} \vee \forall\mathbf{Q}$.
(n) Let $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Pos}$ be the contravariant powerset functor. Construct a natural isomorphism $T : [-; \mathbb{B}] \xrightarrow{\cong} \mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Pos}$.

2.6. **Problem 6.** Consider the *dual functor* $\mathbf{Vect}_k(-, k) : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$. We write $V^* = \mathbf{Vect}_k(V, k)$ for the *dual space* of V. Suppose V has a basis $\mathcal{E} = (\mathbf{e}_i)_{i=1}^n$.

(a) Show that V^* has a basis $\mathcal{E}^* = (\mathbf{e}^i)_{i=1}^n$ given by $\mathbf{e}^i(\mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
(b) As usual, the basis vectors $(\mathbf{e}_i)_{i=1}^n$, when expressed via the coordinates they induce, can be expressed as standard column vectors:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Show that the vectors $(\mathbf{e}^i)_{i=1}^n$, when conceived of as linear maps $V \to k$ in the basis $\mathcal{E}$ can be expressed as standard row vectors:

$$\mathbf{e}^1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \qquad \mathbf{e}^2 = \begin{bmatrix} 0 & 1 & \dots & 0 \end{bmatrix} \quad \cdots \quad \mathbf{e}^n = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}$$

(c) Prove that the linear map $\xi_V : V \to V^* :: \mathbf{e}_i \mapsto \mathbf{e}^i$ is an isomorphism for all V.
(d) Although this is perhaps unintuitive, ξ_V does not extend to a natural transformation $\mathbb{1}_{\mathbf{Vect}_k} \to (-)^*$. Consider the map $L : k^2 \to k^2 :: (a, b) \mapsto (2b, a)$. Show that the following naturality diagram fails to commute.

$$\begin{array}{ccc} k^2 & \xrightarrow{\varphi_{k^2}} & (k^2)^* \\ {\scriptstyle L}\big\downarrow & & \big\downarrow{\scriptstyle (L^*)^{-1}} \\ k^2 & \xrightarrow[\varphi_{k^2}]{} & (k^2)^* \end{array}$$

(e) In contrast, there is a map $\zeta_V : V \to V^{**} :: v \mapsto \lambda\varphi.\varphi(v)$ to the *double dual* $V^{**} = \mathbf{Vect}_k(V^*, k)$. Show that this map extends to a natural transformation.
(f) Prove that for any V, $\ker \zeta_V = 0$. Conclude that $\zeta : \mathbb{1}_{\mathbf{Vect}_k} \to (-)^{**}$ is a natural isomorphism. You may assume that $\dim V = \dim V^{**}$.