3. Assignment 3

3.1. **Problem 1.** Let $\mathcal{C}$ be a category with terminal object $\star$. Define $\widetilde{\mathcal{C}}$ to have objects $(f, x_0) \in \mathcal{C}(X, X) \times \mathcal{C}(\star, X)$ with arrows $(f, x_0) \to (g, y_0)$ given by $\mathcal{C}$-arrows $\varphi : X \to Y$ such that the following two diagrams commute.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[d, "x_0"'] \arrow[dr, "y_0"] & \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

(a) Prove that $\widetilde{\mathcal{C}}$ is a category.

(b) Show that the objects of $\widetilde{\mathbf{Set}}$ can be seen as *discrete dynamical systems*

$$\begin{cases} x_{n+1} = f(x_n) \\ x_0 \end{cases}$$

(c) Define the *successor* map as $s : \mathbb{N} \to \mathbb{N} :: n \mapsto n + 1$. Prove that $(s, 0)$ is the initial object in $\widetilde{\mathbf{Set}}$.

3.2. **Problem 2.** Characterize the terminal object of $\mathcal{C}/c$ and initial object of $c/\mathcal{C}$.

3.3. **Problem 3.** Let $\mathcal{C}$ be a category with products such that $\mathcal{C}^{\mathrm{op}} = \mathcal{C}$.

(a) Prove that $\mathcal{C}$ has biproducts; i.e. that it has coproducts and that these are isomorphic to products.

(b) Recall the category **Rel** consisting of objects sets and arrows $X \to Y$ subsets $R \subseteq X \times Y$, i.e. maps $X \times Y \to \mathbb{B}$. Writing xRy for $R(x, y) = \top$, we compose relations $R : X \to Y$ and $Q : Y \to Z$ by defining $x(QR)z$ if and only if there exists y such that xRy and yQz. Prove that the disjoint union $X + Y$ gives a product in this category.

(c) Prove that $\mathbf{Rel} = \mathbf{Rel}^{\mathrm{op}}$. Conclude that $+$ gives a biproduct in **Rel**.

(d) This allows us to conceive of any relation as a matrix of relations between singletons $\{x\}R\{y\}$. What should the entries of this matrix look like?

(e) Interpret what matrix multiplication should mean in this context.

3.4. **Problem 4.** Show that any equalizer is a monomorphism. Argue by duality that any coequalizer is an epimorphism.

3.5. **Problem 5.** Let $\mathcal{C}$ be a category with pullbacks and a terminal object $\star$.

(a) Let X, Y be objects. Show that their product $X \times Y$ is isomorphic to the pullback of the following diagram.

$$\begin{tikzcd} & X \arrow[d, dashed, "\exists!\varphi"] \\ Y \arrow[r, dashed, "\exists!\psi"'] & \star \end{tikzcd}$$

(b) Let $f, g : X \to Y$ and consider the following pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Prove that the equalizer of f, g is the pullback of the following diagram.

$$\begin{tikzcd} & X \times_Y X \arrow[d, "(\sigma_1,\sigma_2)"] \\ X \arrow[r, "(\mathbb{1}_X,\mathbb{1}_X)"'] & X \times X \end{tikzcd}$$

Conclude that the existence of pullbacks and a terminal object implies the existence of products and equalizers. Now suppose $\mathcal{C}$ has products and equalizers.

(c) Consider the following diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Show that the pullback of this diagram is isomorphic to the equalizer of the following pair of parallel arrows.

$$\begin{tikzcd} X \times Y \arrow[r, shift left, "\pi_X \mathbin{/\!\!/} f"] \arrow[r, shift right, "\pi_Y \mathbin{/\!\!/} g"'] & Z \end{tikzcd}$$

(d) Suppose $\mathcal{C}$ has a nullary product, i.e. (as was argued before) an object $*$ such that $* \times X \cong X \cong X \times *$. Prove that $*$ is the terminal object of $\mathcal{C}$.

(e) Conclude that a category $\mathcal{C}$ has pullbacks and a terminal object if and only if it has products and equalizers. Use duality to argue why $\mathcal{C}$ has pushouts and an initial object if and only if it has coproducts and coequalizers.

3.6. **Problem 6.** This problem is a classic. Consider the following commutative rectangle, whose righthand square is a pullback. Prove that the left hand square is a pullback if and only if the whole rectangle is a pullback.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

3.7. **Problem 7.** Let $\overline{\mathbb{N}} = \mathbf{T}(\mathbb{N}, \leq)$. Consider the diagram $\mathcal{G} : \overline{\mathbb{N}} \to \mathbf{Set}$ defined by $\mathcal{G}n = \mathbf{1} + X + X^2 + \cdots + X^n$ and $\mathcal{G}[n \to n+1]$ the canonical inclusion map $\mathbf{1}+X+X^2+\cdots+X^n \to \mathbf{1}+X+X^2+\cdots+X^n+X^{n+1}$. We show $\operatorname{colim}\mathcal{G} = \operatorname{List} X$.

(a) Show that the maps

$$\iota_n : \mathbf{1} + X + \cdots + X^n \to \operatorname{List} X :: (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n],$$

where $\iota_0 : \mathbf{1} \to \operatorname{List} X :: 0 \mapsto [\,]$, define a cocone to $\operatorname{List} X$. In other words, check the necessary commutativity requirements.

(b) Now suppose there is some other cocone with components $\varphi_n : \mathbf{1}+\cdots+X^n \to Z$. Define a map $\Phi : \operatorname{List} X \to Z$ for which $\iota_n \mathbin{/\!\!/} \Phi = \varphi_n$.

3.8. **Problem 8.** Let $(P, \preceq)$ be a preorder whose associated thin category is denoted by $\overline{P}$. Let $x \in P$ and consider the hom functor $\overline{P}(x, -) : P \to \mathbf{Set}$. This takes values either $\star$ or $\varnothing$ and hence can be reconceptualized as a monotone map:

$$\overline{P}(x, -) : P \to \mathbb{B}.$$

(a) Show that $\mathcal{U}_x = \overline{P}(x, -)^*(\top)$.

(b) In this context, the Yoneda Lemma states:

$$\mathbf{Pre}(\overline{P}(x,-),\overline{P}(y,-)) \cong \overline{P}(y,x)$$

Use this, in conjunction with (a), to prove

$$\mathcal{U}_x \subseteq \mathcal{U}_y \Rightarrow y \preceq x.$$

(c) Now suppose $\overline{P}(x,-) \cong \overline{P}(y,-)$. Show that this implies $x \cong y$.