4. Assignment 4

Recall that an adjunction consists of a pair of functors $(L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C})$, for which we have the following natural isomorphism:

$$\mathcal{D}(Lc, d) \cong \mathcal{C}(c, Rd)$$

We call L and R the left and right adjoints, and write $L \dashv R$ for shorthand.

4.1. **Problem 1.** We will show that there is a length four sequence of adjunctions

$$\pi_0 \dashv \mathbf{D} \dashv U \dashv \mathbf{I}$$

where the *underlying set* functor $U : \mathbf{Cat} \to \mathbf{Set}$ takes a small category $\mathcal{C}$ to its object set $|\mathcal{C}|$ and the *discrete category* functor $\mathbf{D} : \mathbf{Set} \to \mathbf{Cat}$ map a set S to the category $\overline{S}$ whose object set is S and whose only morphisms are identity arrows.

(a) Prove that $\mathbf{D} \dashv U$. For simplicity, leave one variable fixed and simply prove naturality in the other variable.
(b) Give the rule for the behavior of the *indiscrete category* functor $\mathbf{I} : \mathbf{Set} \to \mathbf{Cat}$ so as to satisfy the adjunction isomorphism. You need not check naturality.
(c) The *connected components* functor $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ takes a category $\mathcal{C}$ and sends it to the set of *connected components* of $\mathcal{C}$. More precisely, for $\mathcal{C}$-objects x, y let $x \sim y$ if there exists some *zig-zag* of arrows:

$$x \to t_1 \leftarrow t_2 \to t_3 \leftarrow \cdots y$$

The idea is that when $x \not\sim y$, x and y are intuitively disconnected. We then define $\pi_0\mathcal{C} = |\mathcal{C}|/\sim$. Show that $\pi_0 \dashv \mathbf{D}$. You need not check naturality.

Recall that a monad (T, μ, η) in a 2-category $\mathfrak{B}$ is a monoid object in the monoidal hom-category $\mathfrak{B}(x, x)$ for some object x.

4.2. **Problem 2.** Let $(T : \mathcal{C} \to \mathcal{C}, \mu : T^2 \to T, \eta : \mathbb{1}_{\mathcal{C}} \to T)$ be a monad in $\mathbf{Cat}$. Define the *Kleisli category* $\mathcal{C}_T$ has object set $|\mathcal{C}|$ but arrows defined as:

$$\mathcal{C}_T(x, y) = \mathcal{C}(x, Ty).$$

The composition of arrows in this category then amounts to mapping a pair of $\mathcal{C}$-arrows $f : x \to Ty, g : y \to Tz$ to a $\mathcal{C}$-arrow $x \to Tz$, which we define as the following composition:

$$x \xrightarrow{f} Ty \xrightarrow{Tg} T^2z \xrightarrow{\mu_z} Tz$$

(a) Prove that this composition is associative.
(b) Prove that the unit map $\eta_x : x \to Tx$ defines an identity arrow in $\mathcal{C}_T$.
(c) We secretly use the Kliesli category for the powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ when solving elementary algebra equations. For example, consider the equation

$$\sin^2(x) = 1$$

We first take the preimage of 1 under $\square^2$ to get $\sin(x) = \{1, -1\}$, and then we apply the preimage of sin to *each* of the elements in $\{1, -1\}$ and take the union of the two results. Explain how this procedure instantiates composition of arrows in the Kleisli category.

For the following problem, use diagrammatic methods. Feel free to hand-draw your diagrams and then scan them in as an attachment to your email.

4.3. **Problem 3.** A dual pair (L, R) in a 2-category $\mathfrak{B}$ is a pair of 1-arrows $(L : x \to y, R : y \to x)$, equipped with 2-arrows

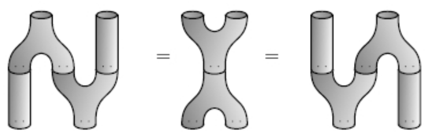
$$\epsilon : L \circ R \to \mathbb{1}_y \qquad \eta : \mathbb{1}_x \to R \circ L$$

satisfying the zig-zag equations.

(a) Prove that $R \circ L$ is a monad with unit η and multiplication given by

$$\mathbf{1}_R \circ \epsilon \circ \mathbf{1}_L : R \circ L \circ R \circ L \to R \circ L.$$

(b) We say that $(X, \mu, \eta, \delta, \epsilon)$ is a Frobenius monoid if (X, μ, η) is a monoid, (X, δ, ϵ) is a comonoid, and the following *Frobenius Law* is satisfied.



Prove that X is self dual by constructing dual pair arrows

$$I \to X \otimes X \to I$$

via composing some arrows in the Frobenius structure, and then showing that they satisfy the zig-zag axioms.